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Quantum $E(2)$ groups and Lie bialgebra structures

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Abstract. Lie bialgebra structures on $e(2)$ are classified. For two Lie bialgebra structures which are not coboundaries (i.e. those which are not determined by a classical r-matrix) we solve the cocycle condition, find the Lie–Poisson brackets and obtain quantum group relations. There is one to one correspondence between Lie bialgebra structures on $e(2)$ and possible quantum deformations of $U(e(2))$ and $E(2)$.

1. Introduction

Quantum deformations [1, 2] of the $D = 2$ Euclidean group $E(2)$ and its universal enveloping algebra $U(e(2))$ turn out to be a useful laboratory for the study of various aspects of quantum groups [3–5]. It is one of the simplest examples of non-simple Lie group and there is no canonical way to introduce its deformation. During the last five years many approaches have been developed [6, 7] for the construction of such deformations. The study of $E_q(2)$ is interesting for its own sake (one can ask questions about how many different quantum deformations exist in this case, about classical r and quantum R matrices, differential calculi, representations, bicrossproduct structures, etc), but it is also useful in order to understand properties of quantum deformations of $D = 4$ Poincaré group [8]. The structure of $D = 4$ quantum groups are important to explore if one wishes to examine possible implications of quantum groups ideas in physics. Interesting results have recently been obtained in this direction, including a classification of possible deformations of the $D = 4$ Poincaré group [9].

The aim of this paper is to argue that all the possible quantum deformations of $E(2)$ can be deduced from the analysis of Lie bialgebra structures on $e(2)$. The paper is organized as follows. In section 2 we review obtained so far in the literature quantum deformations of $E(2)$ and $U(e(2))$. In section 3 we present the classification of the Lie bialgebra structures for $e(2)$. We obtain one one-parameter family of Lie bialgebra structures and three separate 'points'. Some of them turn out to be coboundaries (i.e. they are determined by classical r-matrices) but some are not of that kind. We show that quantum deformations described in section 2 give rise to all of them, except for one case. The missing quantum deformation of $E(2)$ turns out to be the simplest one, and will be discussed in section 4. There we describe in detail how to derive Lie–Poisson brackets corresponding to Lie bialgebra structures which are not coboundaries. In the new case of Lie Poisson brackets that have not yet appeared in the literature, we obtain quantum group relations by changing Poisson brackets

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into commutators. We also calculate by duality the corresponding quantum deformation of $U(e(2))$. In section 5 we conclude the paper with some final remarks.

In our presentation we will concentrate on the algebra and coalgebra structures of $E_q(2)$ and $U_q(e(2))$. It is a trivial exercise to guess what form of antipode and counit make them Hopf algebras.

2. Quantum deformations of $E(2)$ and $U(e(2))$

The first papers of interest were dedicated to quantum deformations of the enveloping algebra $U(e(2))$ [3]. These deformations were obtained by applying the technique of contraction from the standard deformation of $U(sl(2))$. It turns out that there are two different quantum contractions [10]:

(A)
$$[P_+, P_-] = 0 \qquad [J, P_\pm] = \pm P_\pm \tag{1}$$

$$\Delta(J) = J \otimes 1 + 1 \otimes J \tag{2}$$

$$\Delta(P_\pm) = P_\pm \otimes q^J + q^{-J} \otimes P_\pm \tag{3}$$

(B)
$$[P_1, P_2] = 0 \qquad [J, P_1] = \mathrm{i}P_2 \tag{4}$$

$$[J, P_2] = -\frac{\mathrm{i}\kappa}{2}\,\mathrm{sh}\,\frac{2P_1}{\kappa} \tag{5}$$

$$\Delta(J) = J \otimes \mathrm{e}^{-P_1/\kappa} + \mathrm{e}^{P_1/\kappa} \otimes J \tag{6}$$

$$\Delta(P_1) = P_1 \otimes 1 + 1 \otimes P_1 \tag{7}$$

$$\Delta(P_2) = P_2 \otimes \mathrm{e}^{-P_1/\kappa} + \mathrm{e}^{P_1/\kappa} \otimes P_2\,. \tag{8}$$

The deformation parameters are q in case (A) and κ in case (B) and the classical limits are $q \to 1$ and $\kappa \to \infty$. It should perhaps be mentioned that till now there has been no general theory of contractions of quantum groups. Some recent papers [11, 12] investigate such contractions by analysing the Lie bialgebra level.

The quantum group $E_q(2)$ has been discussed by many authors from different points of view [3–6]. In order to fix notation let us introduce the following matrix representation of elements of $E(2)$:

$$g(c, a, b) = \begin{pmatrix} \cos(c) & \sin(c) & a \\ -\sin(c) & \cos(c) & b \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

The matrix multiplication defines coproduct and antipode for a, b and c. It turns out to be convenient to introduce the complex notation

$$\eta = a + \mathrm{i}b \qquad \bar{\eta} = a - \mathrm{i}b \qquad \mathrm{e}^{\mathrm{i}c}\,. \tag{10}$$

The coproducts take the form

$$\Delta(\eta) = \mathrm{e}^{-\mathrm{i}c} \otimes \eta + \eta \otimes 1 \tag{11}$$

$$\Delta(\bar{\eta}) = \mathrm{e}^{\mathrm{i}c} \otimes \bar{\eta} + \bar{\eta} \otimes 1 \tag{12}$$

$$\Delta(\mathrm{e}^{\mathrm{i}c}) = \mathrm{e}^{\mathrm{i}c} \otimes \mathrm{e}^{\mathrm{i}c}\,. \tag{13}$$

There are many approaches for obtaining the quantum group relations for $E(2)$ [4–7]. They lead to two sets of relations:

(A′)
$$\eta\bar{\eta} = q^2\bar{\eta}\eta \tag{14}$$

$$\eta \mathrm{e}^{\mathrm{i}c} = q^2 \mathrm{e}^{\mathrm{i}c}\eta \tag{15}$$

$$\bar{\eta} \mathrm{e}^{\mathrm{i}c} = q^2 \mathrm{e}^{\mathrm{i}c}\bar{\eta} \tag{16}$$

or

(B′)
$$[\mathrm{e}^{\mathrm{i}c}, \eta] = \frac{1}{\kappa}(1 - \mathrm{e}^{\mathrm{i}c}) \tag{17}$$

$$[\mathrm{e}^{\mathrm{i}c}, \bar{\eta}] = \frac{1}{\kappa}(\mathrm{e}^{2\mathrm{i}c} - \mathrm{e}^{\mathrm{i}c}) \tag{18}$$

$$[\eta, \bar{\eta}] = \frac{1}{\kappa}(\bar{\eta} + \eta)\,. \tag{19}$$

The coproducts for η, $\bar{\eta}$ and $\mathrm{e}^{\mathrm{i}c}$ are given in (11)–(13).

It should be mentioned that the full Hopf algebra duality has been demonstrated for two deformations of the groups $E(2)$ and $U(e(2))$ so far discussed. For case (A) and (A′) this was done in [5] and for case (B) and (B′) in [6].

There is yet another approach to the quantization of $E(2)$. The starting point is the non-standard (sometimes called Jordanian) quantum deformation of $U(sl(2))$. Following the general ideas it is possible to perform the contraction from $U_q(sl(2))$ to $U_\mu(e(2))$ [13]. The new deformation parameter is called μ.

(C)
$$[P_+, P_-] = 0 \qquad [J, P_+] = \mu\,\mathrm{sh}\,\frac{P_+}{\mu} \tag{20}$$

$$[J, P_-] = -P_-\,\mathrm{ch}\,\frac{P_+}{\mu} \tag{21}$$

$$\Delta(P_+) = P_+ \otimes 1 + 1 \otimes P_+ \tag{22}$$

$$\Delta(J) = J \otimes \mathrm{e}^{P_+/\mu} + \mathrm{e}^{-P_+/\mu} \otimes J \tag{23}$$

$$\Delta(P_-) = P_- \otimes \mathrm{e}^{P_+/\mu} + \mathrm{e}^{-P_+/\mu} \otimes P_-\,. \tag{24}$$

The bad feature of this deformation of $U(e(2))$ is that it is strictly speaking a deformation of the complex algebra $e(2)$. This is seen in equations (20)–(24). The operation $J^* = J$, $P_\pm^* = P_\mp$ is not a star operation in the Hopf algebra $U_\mu(e(2))$.

3. Classification of Lie algebra structures for $e(2)$

It is possible to give a complete classification of Lie bialgebra structures for $e(2)$. Let us introduce the $e(2)$ Lie algebra with generators P_1, P_2 and J satisfying the relations

$$[P_1, P_2] = 0 \qquad [J, P_1] = \mathrm{i}P_2 \qquad [J, P_2] = -\mathrm{i}P_1\,. \tag{25}$$

In the classification of Lie-bialgebra structures for $e(2)$ one should take into account its invariance under the following transformations: (i) $J \to J + \mu P_1 + \nu P_2$;

(ii) $P_1 \to \cos\beta P_1 + \sin\beta P_2$, $P_2 \to -\sin\beta P_1 + \cos\beta P_2$; (iii) $P_1 \to \lambda P_1$, $P_2 \to \lambda P_2$. The complete list of Lie bialgebra structures consists of

$$\delta_1(P_1) = sP_1 \wedge J \qquad \delta_1(P_2) = sP_2 \wedge J \qquad \delta_1(J) = 0 \tag{26}$$

$$\delta_2(J) = P_1 \wedge P_2 \qquad \delta_2(P_1) = \delta_2(P_2) = 0 \tag{27}$$

$$\delta_3(P_2) = P_1 \wedge P_2 \qquad \delta_3(J) = P_1 \wedge J \qquad \delta_3(P_1) = 0 \tag{28}$$

$$\delta_4(P_1) = -\mathrm{i}P_1 \wedge P_2 \qquad \delta_4(P_2) = P_1 \wedge P_2 \qquad \delta_4(J) = P_1 \wedge J + \mathrm{i}P_2 \wedge J\,. \tag{29}$$

In (26) there is a one-parameter (s) family of Lie bialgebras. Of the above four possibilities only the last two are coboundaries with the classical r-matrices

$$r_3 = J \wedge P_2 \tag{30}$$

$$r_4 = J \wedge P_1 + \mathrm{i}J \wedge P_2\,. \tag{31}$$

One could also write down a more general form of the classical r-matrices by adding terms $\tau P_1 \wedge P_2 + \gamma(P_1 \otimes P_1 + P_2 \otimes P_2)$. This generalization will however turn out to be inessential if one deduces the form of Lie–Poisson brackets out of r. We find two Lie bialgebra structures which are not a coboundary. It is interesting to stress that the case of $D = 2$ is very particular one. It was shown that for $D \geqslant 3$ all the Lie bialgebra structures of homogeneous groups built from space-time rotations (with arbitrary signature) and translations are coboundaries [14].

It is easy to observe that the deformation (A) of section 2 corresponds to δ_1, (B) to δ_3 and (C) to δ_4. On the other hand, the Lie bialgebra δ_2 does not yet have its quantum group counterpart.

4. The missing quantum deformation of $E(2)$

When a Lie bialgebra is not a coboundary the computation of Lie–Poisson brackets is not straightforward. The problem is to solve the cocycle equation for $\phi : G \to \mathcal{G} \wedge \mathcal{G}$, where G and $\mathcal{G}$ respectively denote the Lie group and its Lie algebra [15]:

$$\phi(gh) = \phi(g) + g\phi(h)g^{-1}\,. \tag{32}$$

Let us first discuss the case of the Lie bialgebra structure δ_1. The 'initial' conditions are

$$\phi(1 + \epsilon P_1 + \cdots) = \epsilon s P_1 \wedge J + \cdots \tag{33}$$

$$\phi(1 + \epsilon P_2 + \cdots) = \epsilon s P_2 \wedge J + \cdots \tag{34}$$

$$\phi(1 + \epsilon J + \cdots) = 0 \tag{35}$$

where ϵ is an infinitesimal parameter.

It is understood that the elements of $e(2)$ and $E(2)$ are given in the three-dimensional representation, and that

$$P_1 = \mathrm{i}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P_2 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad J = \mathrm{i}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

The strategy is to find first ϕ on 1-parameter subgroups generated by P_1, P_2 and J. Let

$$\phi(\mathrm{e}^{-\mathrm{i}aP_1}) = A(a)P_1 \wedge J + B(a)P_2 \wedge J + C(a)P_1 \wedge P_2\,. \tag{37}$$

The cocycle equation (32) gives rise to the following set of algebraic equations:

$$2A(a) = A(2a) \tag{38}$$

$$2B(a) = B(2a) \tag{39}$$

$$2C(a) - aA(a) = C(2a)\,. \tag{40}$$

We assume further that the functions $A(a)$, $B(a)$ and $C(a)$ are analytic in a. Taking into account (33) one obtains

$$\phi(\mathrm{e}^{-\mathrm{i}aP_1}) = \mathrm{i}s\left(-aP_1 \wedge J + \frac{a^2}{2} P_1 \wedge P_2\right)\,. \tag{41}$$

Using the same methods one also calculates

$$\phi(\mathrm{e}^{-\mathrm{i}bP_2}) = \mathrm{i}s\left(-bP_2 \wedge J + \frac{b^2}{2} P_1 \wedge P_2\right) \tag{42}$$

$$\phi(\mathrm{e}^{\mathrm{i}cJ}) = 0\,. \tag{43}$$

The group elements of $E(2)$ can be parametrized by

$$g(a,b,c) = \mathrm{e}^{-\mathrm{i}aP_1}\mathrm{e}^{-\mathrm{i}bP_2}\mathrm{e}^{\mathrm{i}cJ} = \begin{pmatrix} \cos c & \sin c & a \\ -\sin c & \cos c & b \\ 0 & 0 & 1 \end{pmatrix} \tag{44}$$

for which, using (32), one calculates

$$\phi(g(a,b,c)) = is\left(-aP_1 \wedge J - bP_2 \wedge J + \frac{a^2+b^2}{2} P_1 \wedge P_2\right)\,. \tag{45}$$

Since one knows from the general theory [15] that

$$\{f, g\} = \phi^{ab}\partial_a f \partial_b g \tag{46}$$

the Lie–Poisson brackets for a, b, c follow:

$$\{a, b\} = \frac{\mathrm{i}s}{2}(a^2 + b^2) \tag{47}$$

$$\{a, \cos c\} = \mathrm{i}as \sin c \tag{48}$$

$$\{b, \cos c\} = \mathrm{i}bs \sin c \tag{49}$$

or, in complex notation

$$\{\eta, \bar{\eta}\} = s\eta\bar{\eta} \tag{50}$$

$$\{\eta, \mathrm{e}^{\mathrm{i}c}\} = s\eta\mathrm{e}^{\mathrm{i}c} \tag{51}$$

$$\{\bar{\eta}, \mathrm{e}^{\mathrm{i}c}\} = s\bar{\eta}\mathrm{e}^{\mathrm{i}c}\,. \tag{52}$$

These expressions should be compared with (14)–(16).

Let us apply the same method to the second non-coboundary Lie bialgebra structure δ_2:

$$\delta_2(J) = P_1 \wedge P_2 \qquad \delta_2(P_1) = \delta_2(P_2) = 0\,. \tag{53}$$

Using the above-described technique once more one obtains

$$\phi(g(a,b,c)) = \mathrm{i}cP_1 \wedge P_2\,. \tag{54}$$

After calculating Lie–Poisson brackets it turns out that the only non-vanishing bracket is

$$\{\eta, \bar{\eta}\} = -2c\,. \tag{55}$$

One can check explicitly that it in fact satisfies the required condition

$$\{\Delta(\eta), \Delta(\bar{\eta})\} = \Delta(\{\eta, \bar{\eta}\}) = \Delta(-2c) = -2(c \otimes 1 + 1 \otimes c)\,. \tag{56}$$

Naive quantization seems to be applicable in this case, so that for the quantum relations one obtains

$$[\eta, \bar{\eta}] = \mathrm{i}hc \qquad [\eta, c] = [\bar{\eta}, c] = 0 \tag{57}$$

where by h we denote the deformation (quantization) parameter. It is instructive to find by duality the corresponding quantum deformation of $U(e(2))$. After brief computations one arrives at the following structure:

(D)
$$[J, P_1] = \mathrm{i}P_2 \qquad [J, P_2] = -\mathrm{i}P_1 \qquad [P_1, P_2] = 0 \tag{58}$$

$$\Delta(P_1) = P_1 \otimes 1 + 1 \otimes P_1 \qquad \Delta(P_2) = P_2 \otimes 1 + 1 \otimes P_2 \tag{59}$$

$$\Delta(J) = J \otimes 1 + 1 \otimes J + h(P_1 \otimes P_2 - P_2 \otimes P_1)\,. \tag{60}$$

This is in fact the simplest possible quantum deformation of $U(e(2))$. The antipode and counit are as in the undeformed case.

5. Conclusion

We conclude that a theory of quantum deformations of the $D = 2$ Euclidean group seems to be almost complete. All the Lie bialgebra structures on $e(2)$ can be quantized to Hopf algebras $U_q(e(2))$. There is still one interesting unsolved problem, however. It is unknown whether a universal R-matrix exists for the quantum deformation (4)–(8). In many cases the contraction prescription can be applied to the R-matrix, giving rise to a finite result (usually after some manipulation) [16, 17]. In case (A) discussed above the classical r-matrix does not exist and the same must be true for the universal R-matrix. In case (B) the situation is unclear. Contraction of the universal R-matrix for $U_q(sl(2))$ leads to divergent expressions. However, direct computation shows that, at least up to terms $1/\kappa^{10}$, an expression for R can be found [18]. This R satisfies the condition $R\Delta(a) = \Delta'(a)R$ for all the elements a, but it cannot satisfy the Yang-Baxter equation, as the classical r-matrix satisfies only the modified classical Yang-Baxter equation.

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